DOCUMENT RESUME

ED 434 859 SO 030 876

AUTHOR Kalchman, Mindy; Katz, Steven
TITLE Symbol or Experience: A Mathematics Education Quandary Illustrated in the Context of Functions.
PUB DATE 1999-04-00
NOTE 26p.; Paper presented at the Annual Meeting of the American Educational Research Association (Montreal, Quebec, Canada, April 19-23, 1999).
PUB TYPE Reports - Research (143) -- Speeches/Meeting Papers (150)
EDRS PRICE EDRS Price MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS *Classroom Techniques; Comparative Analysis; Educational Research; Grade 11; Grade 6; High Schools; Intermediate Grades; Laboratory Schools; *Mathematical Concepts; *Mathematics Education; Metaphors; *Problem Solving; *Symbols (Mathematics)
IDENTIFIERS Cognitive Strategies

ABSTRACT

The tension between providing opportunities for mathematical understandings and the need to teach widely-used notations and algorithms is explained in terms of a folk pedagogical conflict. Divergent pedagogical practices are shown to follow from different mental and epistemological assumptions and manifest themselves in equally divergent student problem solving strategies. Drawing on a popular Darwinian metaphor, algorithms and notations are interpreted as mathematical symbols that are meaningful only if they stand for a set of experiences. This relationship, which forms the basis of an experimental functions curriculum, is contrasted with the common pedagogical practice of substituting the symbol for the experience. The impact of the latter is illustrated by the poor performance of grade 11 students on a functions problem, while the former is argued to contribute to the success of grade 6 students on the same item. Contains 2 figures, a table of data, and 42 references. (Author/BT)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED 434 859

Symbol or Experience: A Mathematics Education Quandary Illustrated in the Context of Functions

Mindy Kalchman Steven Katz

Department of Human Development and Applied Psychology
Ontario Institute for Studies in Education/University of Toronto
252 Bloor Street West
Toronto, Ontario M5S 1V6

SO 030 876

PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL IN MICROFICHE AND IN ELECTRONIC MEDIA FOR ERIC COLLECTION SUBSCRIBERS ONLY HAS BEEN GRANTED BY

Mindy Kalchman

2A TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☑ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

Paper presented at the annual meeting of the American Educational Research Association, Montreal, P.Q., 1999.

BEST COPY AVAILABLE

Abstract

The tension between providing opportunities for mathematical understandings and the need to teach widely used notations and algorithms is explained in terms of a folk pedagogical conflict. Divergent pedagogical practices are shown to follow from different mental and epistemological assumptions and manifest themselves in equally divergent student problem solving strategies. Drawing on a popular Deweyan metaphor, algorithms and notations are interpreted as mathematical symbols that are meaningful only if they stand for a set of experiences. This relationship, which forms the basis of an experimental functions curriculum, is contrasted with the common pedagogical practice of substituting the symbol for the experience. The impact of the latter is illustrated by the poor performance of Grade 11 students on a functions problem, while the former is argued to contribute to the success of Grade 6 students on the same item.

Introduction

Consider the following mathematics problem:

Can you think of a function that would cross the function seen in the graph below? What is the equation of the function you thought of?

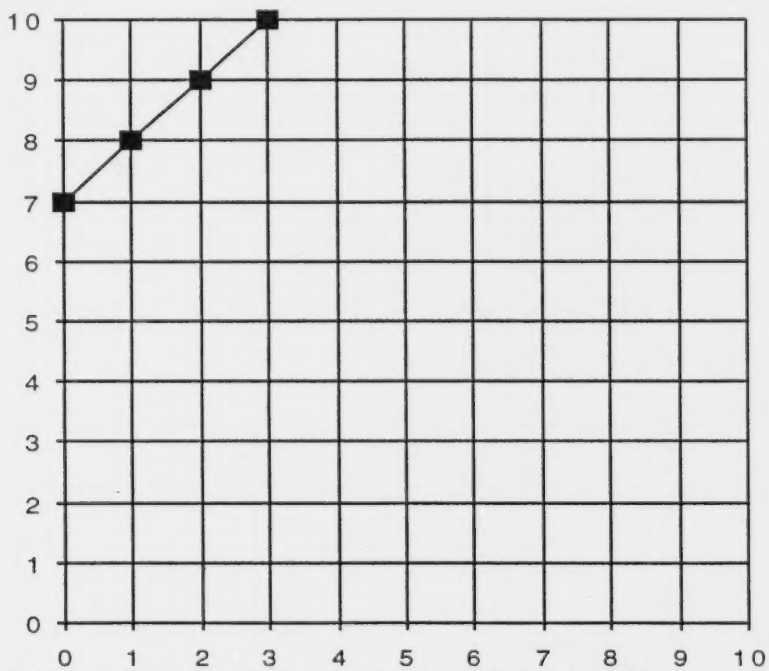

Figure 1 Item presented to Grade 11 and Grade 6 students.

The item is one that invites a range of possible solutions, but all require an understanding of mathematical functions. The topic of functions is arguably one of the most pivotal that children encounter in their secondary school years (Artigue, 1992; Confrey & Doerr, 1996; Eisenberg, 1992; Harel & Dubinsky, 1992). Unfortunately, students of all ages, including post-secondary, find the concepts difficult to master (Artigue, 1992; Harel & Dubinsky, 1992; Leinhardt, Zaslavsky, and Stein, 1990; Sierpinska, 1992). Such a gap in mathematical understanding impedes students' progress with respect to pursuing more advanced mathematics, and selecting career paths that

require application of the function concept. By Grade 11, students have been exposed to functions for close to three years and are deemed psychologically and pedagogically able to answer the aforementioned problem (Case, 1997, 1998; Kalchman & Case, in press; Costa, personal communication; Dunfield, personal communication). However, when recently presented to a Grade 11 class only 9% of the students who received the test item proffered a correct solution.

In this paper, we present an error typology of the incorrect responses to the item in support of what we see as a more general problem in mathematics education. The tension that exists between providing opportunities for mathematical understanding on the one hand, and the need to teach widely used notations and algorithms on the other, is explained in terms of a folk pedagogical conflict (Olson & Bruner, 1996; Olson & Katz, in press). Drawing on a popular Deweyan metaphor (1902/1966), algorithms and notations are interpreted as mathematical symbols that are meaningful only if they stand for a set of experiences. The appeal of the metaphor lies in its prescription of a pedagogy which both recognizes and relates symbols to experiences. However, pedagogical practices in mathematics education are argued to be generally characterized by teacher-determined expository activities aimed at substituting the symbol for the experience. This, as we will see in the Grade 11 context, results in erroneous attempts to remember and manipulate formulae and notations in the absence of experiential understanding. In contrast, we present an experimental functions curriculum that proceeds in accordance with the Deweyan metaphor and provides evidence for its efficacy by reference to a 100% success rate on the above noted item in a *Grade 6* class.

Equating The Symbol And The Experience: A Pedagogical Fallacy Exposed

The mathematics education literature is replete with examples documenting the tension that exists between providing opportunities for mathematical understanding on the one hand, and the need to teach widely used notations and algorithms on the other (Burns, 1994; Carraher & Schliemann, 1985; Kamii & Dominick, 1997; McNeal, 1995; Vakali, 1984). For example, multiplying 86 by 3, for most, would likely involve the following set of procedural rules: (1)

multiply 6 by 3 to get 18; (2) "put down" the 8 and "carry" the 1; (3) multiply 8 by 3 to get 24; (4) add the "carried" 1 and "put down" 25. Others, such as Magdalene Lampert (1986), believe that learning multi-digit multiplication requires an understanding of multiplication as a form of addition that involves principles of regrouping. In her program (see Lampert, 1986 for details), exercises involving "parting and wholing" replace the transmitted set of procedural rules given above. Conceptualizing the differences between the two approaches as a matter of teaching technique masks the fact that important differences occur beneath the level of classroom practice. Underlying the two pedagogical approaches are different conceptions of what it means to learn multiplication, of what it means to learn mathematics, of what it means, more generally, to learn. The differences, it can be argued, exist at the level of what we may call a folk pedagogy - a cohesive set of beliefs about learners' minds and knowledge that guide and structure our pedagogical practices (Olson & Bruner, 1996; Olson & Katz, in press).

Two quite different folk pedagogical models characterize the contours of the current educational landscape, each of which is best conceptualized in metaphorical terms (Earl & Katz, in press). In the "container" tradition, learners' minds are considered devoid of the necessary facts, rules and principles which must then be transferred in a unidirectional fashion from the instructor, text, or other resource to the student. Teaching, therefore, becomes an exercise in telling, while learning is an exercise in remembering.

If teaching is telling and learning is remembering, it follows that something must be told and something must be remembered. That something is knowledge, and for it to take on the status of a transferable commodity it must achieve an existence independent of the individual knower. Knowledge, considered in this light, is certain and permanent (Chandler, Boyes, & Ball, 1990; Kuhn, 1991, 1992; Perry, 1970). It lies with, and is acquired from, the appropriate authorities (Kitchener, 1983; Kitchener & King, 1981; Kuhn, 1992; Perry, 1970). Many educators have experienced this in its purest form when, upon asking a student for an explanation, received the reply , "Because you said so".

In contrast to the container view of mind, the "mind as constructor' metaphor acknowledges the child's mind as an active interpreter of the external world. Children are seen as individuals capable of sense-making both on their own and through discourse with others. Rather than ignoring the privately held beliefs and ideas (Olson & Bruner, 1996; Olson & Katz, in press) of the child, they are brought into a shared frame of reference through the teacher's actions as a facilitator. Thus, in coordinating perspectives learning becomes an exercise in belief evaluation or what Olson and Bruner (1996) call "knowledge construction". Knowledge emerges from a process of interpretation and clarification of meanings related to various aspects of *experience* in the world (Dewey, 1938).

Let us return to the arena of mathematics education. If we accept the notion of a folk pedagogy as put forth here, we begin to understand the perspectives that shape and direct pedagogical practice. Equating mathematics learning with the acquisition of teacher transmitted procedural rules provides direction and sets constraints. Subscribing to the "mind" as container metaphor lends credence to the "put down the ___ and carry the ___" teaching strategy, but precludes taking the path that Lampert (1986) suggests. To teach multi-digit multiplication as Lampert does is to believe that learners are capable of deriving the unknown from the known. Guided by the facilitating hand and voice of the teacher, students access and build on their knowledge of multiplication as a form of repeated addition, eventually constructing an understanding of multi-digit multiplication.

The situation, as it has been presented thus far, poses somewhat of a quandary for the mathematics educator. The elucidation of competing folk pedagogies seems to indicate the absence of a place for the "traditional" algorithmic procedure in a "constructivist" world of mathematics education. Indeed, many have argued that it should not be there at all (Burns, 1994; Carraher & Schliemann, 1985; Kamii & Dominick, 1997; McNeal, 1995; Vakali, 1984). However, the advance of mathematics is, in part, the invention of notations and algorithms that simplify our tasks, hence the need for students to learn them. How, then, do we proceed? One promising direction comes to

us in the form of a metaphor from the well-known educational philosopher, John Dewey (1859-1952).

In a seminal work, *The Child and the Curriculum*, Dewey (1902/1966) provides the illustration of an explorer blazing trails across new terrain, a journey which culminates in the construction of a map once the territory has been thoroughly explored. Of particular relevance is the necessarily mutually dependent relationship between the journey and the map. The map is a symbol of the experience, of the journey. It does not take the place of the journey, but rather provides an arranged and orderly depiction of it. Its utility, thereafter, lies in its ability to propose economic direction on future journeys.

Applying the Deweyan metaphor to the realm of mathematics education, we begin to expose a pervasive fallacy - that of equating the symbol for the experience instead of preserving an ordinal relationship between the two. The algorithm is a symbol, an efficient notational form that conveniently and economically represents the wanderings of the mathematical community; in other words their experiences. The transmission folk pedagogical form described earlier supports the treatment of the symbol as a meaningful and independent knowledge object. To learn mathematics is to manipulate and remember the symbol. The experience, at best, becomes an optional and unendorsed tangent. That is, opportunities for authentic mathematical discovery, for numerical meaning-making, are rarely presented with a pedagogical intention.

Earlier we saw that subscribing to a container model of mind precludes the advocacy of an experience-centered pedagogy. What though of a constructivist folk pedagogy? Does it negate the value of the symbol in an analogous way? If the answer to this question is yes, then it seems that we have merely succeeded in replacing one limited course of action with another. Fortunately, the answer, as we see it, is not yes. Symbol and experience are, by definition, mutually dependent. Moreover, there is a necessarily ordinal relationship between the two elements, with the symbol following from, and representing, the experience for which it stands. It is the attempt to give the symbol without the experience that forces the relationship into one of artificial exclusivity.

The opportunity to preserve the relationship between symbol and experience is indeed a fortunate one. Symbols, as Dewey (1902/1966) reminds us, are tools by which individuals enter the unexplored. They are the means by which we bring to bear whatever we have gained in past searchings. "But this happens only when the symbol really symbolizes - when it stands for and sums up in shorthand actual experiences which the individual has already gone through. A symbol which is induced from without, which has not been led up to in preliminary activities, is, as we say, a *bare* or *mere* symbol [italics original]; it is dead and barren" (p. 24). Consider the evidence of the pedagogical dangers associated with substituting the symbol for the experience in the context of the functions item presented at the outset of this paper. As we will see, the demand to produce an equation leads students to attempt the recall and manipulation of one of the staple symbols of the high school mathematics curriculum, $y = mx + b$ (the formula for a straight line), in the absence of experiential understanding.

The Item: Solving A Functional Problem

The item under consideration is one that appears on a test measure designed to investigate children's conceptual understanding of mathematical functions (Case, 1997). Two secondary level teachers of mathematics have independently validated this item as one they would expect their students to be able to do successfully following the completion of a functions unit. Furthermore, preliminary investigations suggest that by Grade 11, students are able to correctly answer this question (Case, 1997, 1998; Kalchman & Case, in press).

The only constraint delimiting answers to this question is that students must provide a specific equation that will cross the given line within the Cartesian territory seen on the page. There are an unlimited number of possible solutions for the item, and different ways in which the problem may be approached. Discussion around these will be separated into linear and curvilinear. Regardless of the strategy, success depends on an ability to move flexibly between representations of functions -- a skill that has been reported as a troublesome one (Artigue, 1992; Goldenberg, 1988, 1995; Markovits, Eylon, & Bruckheimer, 1986; Sierpinska, 1992).

Any number of linear functions could satisfy the requirements of the problem. If one wanted the new function to pass through the origin, then the equation must have a slope of approximately 3.5 or greater. For example, y=6x, y=8x, and y=10x would all be correct solutions. A second possibility for an increasing linear function is one with a positive y-intercept less than 7 and a slope greater than 1. Possible solutions include $y = 2x + 4$ or $y = 4x + 1$. Third, a decreasing function with a slope less than -1 and a y-intercept greater than 7 and less than 10 would suffice: Functions such as y=-x+10, $y = -3x + 8$, or $y = -2x + 9$ all exemplify this type of solution. Finally, horizontal lines that cross the given line such as $y = 9$, or $y = 8$ are also correct.

Students can also provide equations that produce curved graphs. These functions may be increasing or decreasing. An increasing curvilinear function must have a rate of change that is rapid enough to cross the given line. The function may be adjusted according to whether or not a y-intercept greater than 0 has been selected. Examples of such functions include $y = x^4$, and $y = x^2+4$. A decreasing curvilinear function must fall from a point above the function seen. Sample correct equations are $y = -x^2+8$ or $y = -x^3+9$.

In an effort to produce a correct response, students employ cognitive strategies that not only reflect their understanding of the problem, but also the pedagogical experiences they have been exposed to in math class. Two strategies that students utilize are (1) equation → concept, which we argue results from a pedagogy that supports the symbolic approach to mathematics, and (2) concept → equation, which we argue results from a pedagogy that supports the experiential approach to teaching and learning mathematics. When detailing how each of these strategies correspond to a specific pedagogy, we will focus on linear functions. This is because traditional instructional sequences concentrate on the algebra and geometry of linear functions, and students are most familiar with manipulating the algebraic and geometric properties of straight lines. Consequently, virtually all of the Grade 11 data is drawn from linear functions.

Students who demonstrate the equation → concept solution strategy are those who have had

consistent exposure to a pedagogy that supports substituting the symbol for the experience. For these students, symbols become their staple mathematical tools. Consequently, since the equation for a straight line constitutes the bulk of what students "know" about functions, all related problems converge on that symbol and students are forced to begin with the generic algebraic expression for a straight line, $y = mx + b$. From that generic symbol, students need to derive a conceptual meaning for the structure of that equation in order to arrive at a specific equation that is consistent with the information presented in the graph.. However, as we will see, many students do not ever get beyond the general equation. We argue that within a pedagogy which promotes the substitution of the symbol for the experience, there exists the assumption that if students can successfully use the symbol, they have an understanding of the underlying concept.

On the other hand, students who employ the concept → equation strategy approach the problem conceptually, and often visually (Eisenberg, 1992), by seeing a graphic solution and then translating that graphic representation into an algebraic expression. Successful execution of this approach requires a comfort and familiarity with the experience of the function concept, from which students can generate a variety of specific as well as general symbolic representations (graphic or algebraic). To achieve this, students must be regularly exposed to a pedagogy which promotes experience with (1) different linear functions, (2) determining how those different functions are represented graphically and algebraically, and (3) discovering the general algebraic pattern into which all linear functions fit, namely $y=mx+b$. Students who use the concept → equation strategy are unknowingly preserving the integrity of the ordinal relationship of experience to symbol as suggested by Dewey (1902/1966) -- a relationship that is implicitly modelled in the supporting pedagogy.

In the remainder of this paper we will illustrate the influence that pedagogy has on problem solving strategies by (1) presenting data pertaining to Figure 1 that was collected following a Grade 11 advanced mathematics class receiving their annual text-based unit on functions; (2) introducing an experimental approach to teaching functions that embraces the experience to symbol pedagogy;

and (3) presenting data on the same item collected from Grade 6 students following their participation in the experimental program.

Grade 11: Equation → Concept

The participants of the grade 11 class (n=33) attended a publicly funded, Catholic secondary school in a major urban centre. The students had been exposed to functions for three years as part of the regular mathematics curriculum. Each eleventh grade student was given the whole test in an interview format after a unit on functions. The regular classroom teacher taught the unit as given by the text used by all mathematics teachers in the school. Students received ten, 65 minute classes. The teacher explained that he followed the unit as laid out in the text, only assigned questions that were printed in the text, and followed up the unit with a test suggested and provided by the publishers of the text.

The instructional sequence for the text-based unit proceeded as follows: Functions were introduced as "input-output" machines, in which quantities are input and transformed into output according to a processing rule given algebraically. Next, a set of graphs with corresponding equations was provided as representative samples of different types of functions. Several lessons were spent compiling and applying definitions related to the function concept (e.g. function, relation, domain, range, slope, intercept, parabola). Exercises were primarily procedural (e.g. creating tables of values, point graphing, and algebraically and graphically determining slope and intercept for given equations). Toward the end of the unit students were challenged to determine one form of representation given another.

The above program proceeds in keeping with a pedagogical approach that substitutes the symbol for the experience. This pedagogy, in which lessons are highly prescribed and students' exercises follow directly from the lessons, is appealing to mathematics educators for several reasons. First, it is an efficient way of fulfilling curricular mandates (Ontario Ministry of Education and Training, 1997). Second, it provides their students with the superficial tools needed for performing well on tests (Skemp, 1976). Third, attention to symbol systems is less cognitively

demanding for the teacher than promoting conceptual understanding (Eisenberg, 1992; Skemp, 1976). As a result, students experience the topic in an almost purely symbolic way and may develop a false sense of what it means to understand mathematics in general and of what it means to understand functions in particular.

Following the unit of instruction, only 3 or 9% of the 33 students gave a correct solution to the item under consideration, all of which were linear. Those solutions were y=10-x, y=-2x+10, and y=8. Of the 91% who did not provide a correct solution, nine students did not respond to the question, leaving 21 incorrect responses. Erroneous solutions were classified as i) presentation and manipulation of the generic formula y=mx+b; ii) y = -x; iii) specific incorrect equations; and iv) anomalous responses. With the exception of a single student, all proffered solutions of a linear sort. One interpretation of this attraction to linearity lies in students' preference for linear functions (Markovits et al, 1986; Swan, 1982). Literature suggests that students often reject non-linear graphs as possible translations from other representations of a function based on limited experiences with non-linear graphs as well as an incomplete understanding of the interaction between the function and the situation it represents (Bell & Janvier, 1981; Geer, 1992; Markovits et al, 1986).

The first type of error to be discussed involves an observed reliance on the generic form of the linear equation y = mx + b. Twenty-four percent of the students provided some rendition of the said formula. Three of those students simply gave "y = mx + b" as their answer. The remaining five students somehow altered the y = mx + b formula to indicate a "downward" function. For example, two students gave the equation y = -mx - b. The other three students gave the equation y = -mx + b.

The students who responded with y = mx + b seemed to simply recall a familiar symbol that they knew to relate to functions. Substituting in specific quantities may have seemed either unnecessary or too difficult for these students. Those who responded with y = -mx - b may have been confused as to how the equation y = mx + b needs to be manipulated in order to produce a line that begins above the origin and slopes down. They likely remembered that negative signs are

somehow involved in producing a decreasing line, but were unsure of how to represent that algebraically. The students who responded with $y = -mx + b$ likely understood that a straight line with a "-m" and "+b" would satisfy the conditions of the problem -- a line with a negative slope and a positive intercept -- but could get no further. All of these students were recalling a familiar symbol and attempting to manipulate it in order to fit the conditions of the problem. However, the lack of specific quantities in each of the aforementioned cases might suggest a lack of meaning associated with the individual letters which constitute the equation. In other words, the equation remained an empty symbol.

Seven students provided $y = -x$ as a response. There are several possible interpretations of this solution. The most obvious seems to be that these students were merely indicating a general understanding of how to symbolically represent a decreasing line. However, a second more complex interpretation of this response may be considered in light of literature that suggests the difficulty that students have in differentiating variables from coordinates, and coordinates from intercepts (Schoenfeld, Smith & Arcavi, 1992). Many students are quite confident in their knowledge that the "b" in the equation $y = mx + b$ stands for the y-intercept. However, they are not sure as to what that means in terms of mapping that understanding into Cartesian space (Moschovich, 1996; Schoenfeld et al, 1992). For students who mistake coordinates for intercepts and confuse x and y variables, it is quite conceivable that the "b" in the $y = mx + b$ could get translated into a 0, since the function $y = -x$ is descending from a place on the y axis where $x = 0$. Further, since students are taught that there is no need to represent the "0" in the equation, the final response is $y = -x$. Either interpretation lends credence to the hypothesis that misunderstanding derives from a lack of meaning associated with mathematical symbols.

Of the remaining six students in this sample, three provided the equation of the function given ($y = x + 7$), two students gave incorrect equations ($y = -2x + 1$, and $y = -x^3$), and one student gave an anomalous solution in which she suggested that the answer would be "a function with the

opposite slope or something weird like that." Table 1 provides a breakdown of the results of the solution analysis.

Table 1

Types of errors made by grade 11 sample

Type of Error	Percentage of students (n = 33)
Correct	9
$y = mx + b$	9
$y = -mx + b$	9
$y = -mx - b / y = mx - b$	6
$y = -x$	21
$y = x + 7$	9
No response	27
Wrong equation	6
Verbal description	3

From Experience to Symbol: An Experimental Curriculum

As we have seen, relying on a symbol-based pedagogy for adequately preparing students to participate in the discourse of mathematics is not ideal. It greatly limits students' ability to think flexibly in mathematical terms, and to form conceptual bridges among the variety of interconnected topics that make up the discipline of mathematics. As Dewey (1902/1966) noted, symbols that are transmitted in the absence of preliminary activities remain empty. Earlier, we raised the question, "What might such preliminary activities that give meaning to symbols look like in the mathematics classroom?" One such model in the context of mathematical functions is an experimental curriculum that capitalizes on children's natural experiences with and intuitive understandings of the topic (Case, 1997; Kalchman, 1998; Kalchman & Case, in press). The program is grounded in established principles of cognitive psychology (e.g. Case, 1996) and pedagogical technology

(Goldenberg, 1988; Hillel, Lee, Laborde & Linchevski, 1992; Magdison, 1992). The instructional sequence proceeds as follows.

The basic function concept is introduced using a familiar situation that draws on students' everyday experiences with functions, namely a Walkathon. Students are familiar with the functional relationship between money and distance, whereby the amount of money one earns depends on the distance walked as defined by a pre-arranged sponsorship agreement.

In the first lesson, a table of values is created to represent the money earned at 0, 1, 2, 3, ... 10 kms under different sponsorship rules. As each row of the table for a given rule is completed, the function is graphed on a large 4' by 5' graph (see figure 2). Next, the properties of different rules of sponsorship are compared in their graphic, tabular, and algebraic forms. The main features compared are: (1) the rate at which money accumulated in the tabular form, (2) the steepness (slope)[1], direction (increasing or decreasing), and starting point (y-intercept) of the line or curve in the graphic form, and (3) the effects of different sponsorship agreements. For example, the rule "Anne earns $2.00 more than each kilometer walked (y=x+2)" is compared with "Anne earns $2.00 times the number of kilometers walked (y=2x), and "Anne earns the value of the number of kilometers walked times itself ($y=x^2$)." Initially, equations are written using both conventional and intuitive symbols. For example, because "money earned" is on the vertical axis and "distance walked" along the horizontal axis, "$" is used for the y values and "km" for the x values. Thus, y = x + 2, would also be written $ = km + 2 (when students begin using the computer technology, they are fluent-in using conventional notation). Finally, students invent new ways to sponsor each other and the consequences of these are also explored using all three forms of representation. Students are not restricted to using linear functions.

---

[1] Terms such as "slope" and "y-intercept" as well as conventional symbolic notation (e.g. y=x+2 and $y = x^2$) were not explicitly introduced in this first lesson.

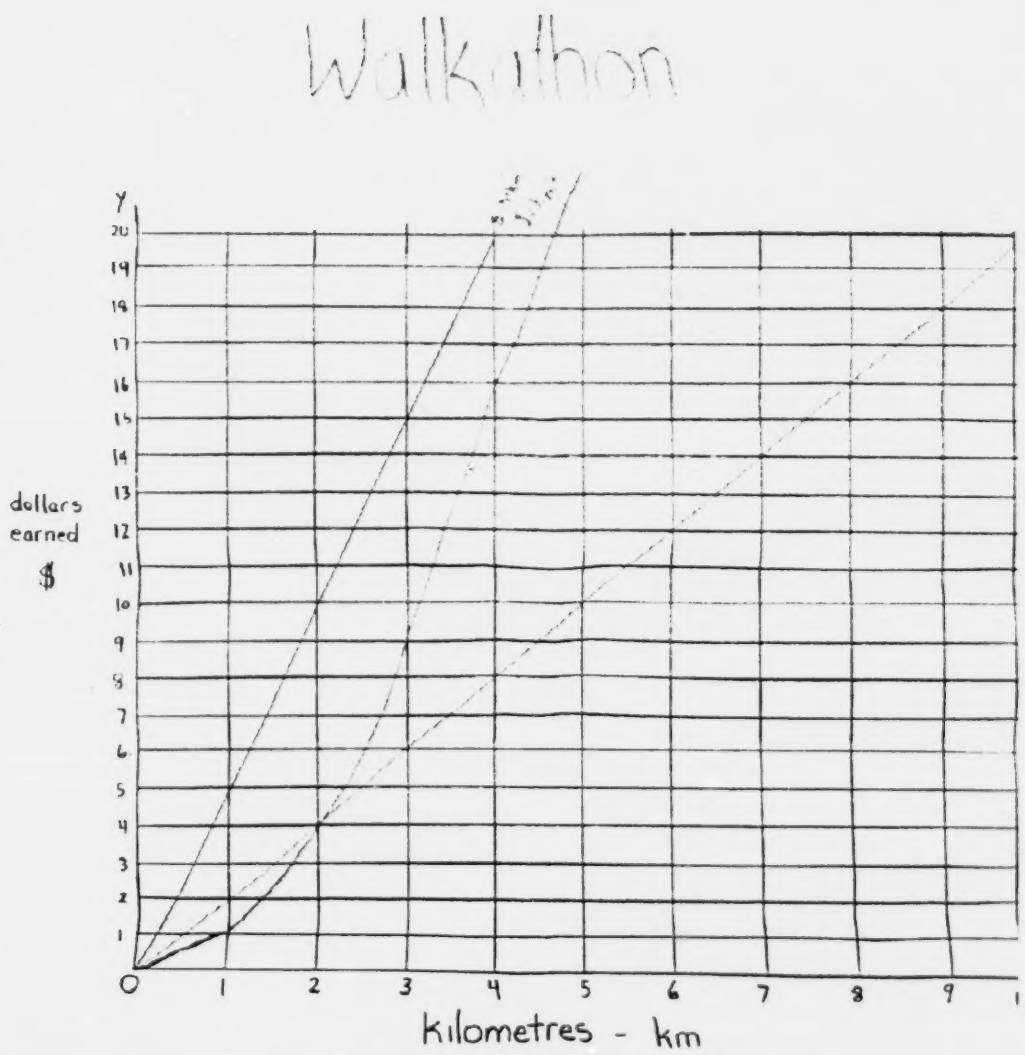

Figure 2 Picture of the 4' x 5' graph completed by students in the first lesson of the experimental curriculum.

In the second lesson there is discussion about the previous day's lesson, specifically about how to get equations from graphs, how the steepness and direction of a graph are changed by changing the rule of sponsorship, and how those changes are reflected in an equation. The idea of an initial donation is introduced and how that donation changes only the "starting place" and not the "steepness" (Moschovich, 1996) of the graph. For example, $100.00 may have been donated by the organization for whom money is being collected, and therefore, that $100.00 must be taken into account when representing the rule of sponsorship in an equation, a graph, and a table of values. So, if the rule of sponsorship is "each kilometre walked times itself, the equation becomes $\$=km^2 + 100$ ($y = x^2 + 100$), the graph has a y-intercept of 100, and the table of values shows a value of 100 at 0 kilometres. Students are challenged to create at least two different linear functions that would provide a formula for raising some specified amount over a 10 km distance. One function was to have an initial donation, and the second was to begin with $0.00.

In the third lesson, students share the functions they invented for earning the specified amount. Curved functions are reviewed and the equation $\$ = km^2$, or $y = x^2$, is reintroduced. Students work on creating curvilinear functions for earning that specified amount that includes both an initial donation and a starting amount of $0.00.

In the next lesson, the notion of negative values along the vertical axis is introduced using the walkathon context. Students devise situations in which negative values would be necessary for this purpose. For example, it has been suggested that something such as a television for a homeless shelter had been purchased on credit, and the amount needed to pay off that debt could be "walked" off in a walkathon. Students work on creating situations and corresponding functions that would necessitate the use of negative values along the vertical axis.

Properties and behaviors of functions are then further solidified in three computer sessions in which pairs of students use a spreadsheet program to manipulate the parameters of functional equations (linear and curvilinear). The consequences of these changes are shown on a graph and in

a table of values which are both instantly and automatically generated by the computer. For example, when exploring the equation y = x + b, students can change the value of "b", and witness the effect that has on both the graph and the table of values. Also, the original linear function is always y = x. Students are able to program a new equation such as y = x - 4, which would be seen on the same grid as y = x, and then compare the equations, graphs, and tables of values in order to evaluate the role that "b" plays in the function as a whole and not just in the equation.

Finally, groups of students spend three days preparing and then delivering a presentation on a specified type of function (e.g. linear, power, exponential growth) about which they become experts during the computer sessions. They share their expertise with classmates, and using graphs, equations, and tables of values, demonstrate what happens when constants and coefficients are changed. Students are also responsible for questioning their classmates about their presentations, and for providing feedback to the presenters (Lampert, Rittenhouse, & Crumbaugh 1996).

Throughout the program, students are immersed in direct experience with notions of sign, slope, intercept, and the relationships among them. Subsequently, conventional symbols such as "m", "b", and "y=mx+b" take on a meaning that would be lacking in the absence of opportunities for students to explore the limits of their understandings about functions on an ongoing basis as they build on their experiences with the function concept. This, as we have seen, is achieved within a framework of organized activities which encourage students to move toward using conventional representations of personal understandings. In the following section, we illustrate how Grade 6 students exposed to this pedagogy adopt the concept → equation strategy when solving the problem seen in Figure 1.

Grade 6: Concept → Equation

The grade six students who participated in the study (n = 10) attended a private university laboratory school in the same urban centre as the Grade 11 sample. Like the Grade 11 participants, these students were also interviewed on an individual basis after instruction. They experienced the experimental functions curriculum in ten, one-hour sessions. The regular classroom teacher and the

present first-author co-taught the unit. Because functions are not normally taught until about grade nine (Leinhardt et al., 1990), students had no experience with the function concept, and thus were not equipped with representative algebraic expressions.

Following the unit of instruction, 100% of the Grade 6 students provided a correct solution to the item under consideration . Though these students were exposed to the generic formula (symbol) for a straight line (y=mx+b) as part of the experimental curriculum, nowhere in their answers was there evidence of their relying on it. Responses were varied with respect to type of function (e.g. linear versus curvilinear) and notational form. For example, answers included $y = x^3 + 5$, $y = x^2 + 6$, $y = x * 8 + 5$, and $y = X_x - 2 + 10$. Of particular interest is the lack of conformity to the conventional structure for an equation of a line. That is, the independent variable often preceded the slope coefficient.

There was evidence that these students were moving from concept to equation, solving the problem conceptually and then representing the solution in algebraic terms. For example, one student gave the equation $y = x * 100$ and supported his response with, "You can just make it times 100 and it'll go straight up." This student was aware of the fact that in the linear sense, in order to obtain a slope large enough to surpass the given function each x value had to be multiplied by some amount that would generate relatively large y values. Using a coefficient of 100 was an exaggeration of the conceptual logic. A second piece of evidence is that one student provided the equation $ = km x 4. This representation directly reflects the walkathon experience, and shows that the experience was translated into a symbol system which is conceptually, albeit not traditionally, correct.

It is interesting to note that 40% of the Grade 6 students chose to give nonlinear solutions versus 3% of the Grade 11 students (in the latter case, one student provided the incorrect equation $y = -x^3$). In the experimental curriculum, the concept of function was emphasized and the symbol system followed from the experience. Consequently, students did not have troubling

translating their understanding of the concept into representative symbols. In fact, one student began with $y = x^2$, and continually redrew the same function, each with a greater and greater y-intercept, until she was satisfied with $y=x^2+6$. This is in stark contrast to the Grade 11 students who typically began with an equation that already had the y-intercept "place" held with a "b", and did not substitute that "b" for a specific quantity.

Conclusions

The function concept is central to virtually all higher-order mathematics. Without an experiential understanding of the topic, students will be unable to participate in the discourse of broader mathematics communities. Such communities transcend professional mathematicians and include engineers, economists, scientists, and teachers of mathematics at all levels. Fostering such an experiential understanding, as we have seen, is often not the dominant pedagogy. Rather, teachers who adhere to a traditional pedagogy and have the structural appearance of an appropriate solution in mind, transmit this expectation directly to their students.

Though relative novices in the domain of functions, sixth grade students outperformed their eleventh grade counterparts in both qualitative and quantitative terms. Through a folk pedagogical lens, we argued that as recipients of the experimental functions curriculum, the former group profited from the ability to preserve an ordinal relationship between experience and symbol, while the latter succumbed to the pervasive pedagogical approach of substituting the symbol for the experience. That is, the Grade 6 students successfully solved the item under consideration by utilizing a cognitive strategy in which they moved from concept to equation. In contrast, the Grade 11 students began with a generic equation and often arrested at this point, failing to arrive at a correct solution. We see this not as an inherent problem with the strategy of moving in the opposite direction (equation to concept), but rather as a result of a transmissive pedagogy that treats symbol and experience as equivalent. As such, students treat equation and concept as analogous and do not to use the former as a precursor for the latter.

It is important to note that we do not advocate for a pedagogy that embraces experience at the expense of symbols. Mathematical algorithms and notations are both efficient and functional, but only when imbued with meaning. Participation in mathematics communities requires fluency with symbols such as $a^2+b^2 = c^2$, $ax^2+bx+c = 0$, $t_n=a+(n-1)d$, $A=\pi r^2$,. and, of course, $y=mx+b$, though not as inert declarative knowledge. We have shown that attention to a pedagogy that respects the ordinal relationship between experience and symbol is one means to this end.

The evidence presented here suggests that careful attention to pedagogical choice (Lampert, 1992) that supports cognitive development (Case, 1996) makes it possible to teach difficult mathematical concepts earlier than is often the case. In fact, pedagogical redefinitions of the sort we have suggested seem preferable earlier rather than later in a child's education. This is because students in their latter years of formal schooling have become comfortable with a definition of learning that follows from the dominant folk pedagogy to which they have been exposed, making it difficult to begin the redefinition process. Therefore, adoption of an alternative folk pedagogy would be more feasible at early stages when students are only beginning to formulate definitions of competence in a particular domain. Research designed to explore this hypothesis is presently underway.

References

Artigue, M. (1992). Functions from an algebraic and graphic point of view: cognitive difficulties and teaching practices. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy. West laFayette, IN: Mathematical Association of America. 109-132.

Bell, A. & Janvier, C. (1981). The interpretation of graphs representing situations. For the Learning of Mathematics, 2(1), 34-42.

Burns, M. (1994). Arithmetic: The last holdout. Phi Delta Kappan, 75, 471-476.

Carraher, T., & Schliemann, A. (1985). Computation routines prescribed by schools: Help or hindrance? Journal for Research in Mathematics Education, 16, 37-44.

Case, R. (1996). Reconceptualizing children's central conceptual structures and their development in middle childhood. In R. Case & Y. Okamoto, The role of central conceptual structures in the development of children's thought. Monographs of the Society for Research in Child Development, 61 (Serial No. 246). 1-26.

Case, R. (1997). Developing children's capabilities for abstract mathematical thought: An experimental program for teaching linear and non-linear functions. Proposal submitted to the J.S. McDonnell Foundation.

Case, R. (1998). A psychological model of number sense and its development. In R. Case (Chair), Fostering the development of number sense in the elementary and middle grades. Symposium conducted at the meeting of the American Educational Research Association, San Diego, CA.

Chandler, M., Boyes, M., and Ball, L. (1990). Relativism and stations of epistemic doubt. Journal of Experimental Child Psychology, 50, 370-395.

Confrey, J. & Doerr, H.M. (1996). Changing the curriculum to improve students understanding of functions. In D.F. Treagust, R. Duit, & B.J. Fraser (Eds.), Improving teaching and learning science and mathematics. New York: Teachers College Press. 162-171.

Dewey, J. (1902/1966). The child and the curriculum. Chicago: University of Chicago Press.

Dewey, J. (1938). Experience and education. New York: Collier.

Earl, L. and Katz, S. (in press). Classroom assessment: Teachers' struggles to change. In N. Bascia and A. Hargreaves (Eds.), The evidence of educational change. London: Falmer.

Eisenberg, T. (1992). On the development of a sense for functions. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy. West laFayette, IN: Mathematical Association of America. 153-174.

Geer, C. (1992). Exploring patterns, relations, and functions. Arithmetic Teacher, 39(9), 19-21.

Goldenberg, E. P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. Journal of Mathematical Behavior, 7,(2), 135-173.

Goldenberg, E. P. (1995). Multiple representations: A vehicle for understanding understanding. In D. Perkins, J. Schwartz, M. Maxwell West, & M. Stone Wiske (Eds.), Software goes to school. New York: Oxford University Press. 155-171.

Harel, G. & Dubinsky, E. (Eds.) (1992). The concept of a Function: Aspects of epistemology and pedagogy. West laFayette, IN: Mathematical Association of America.

Hillel, J., Lee, L., Laborde, C., & Linchevski, L. (1992). Basic functions through the lens of computer algebra systems. Journal of Mathematical Behavior, 11,(2), 119-158.

Kalchman, M. (1998). Developing children's intuitions about mathematical functions in the middle grades. In R. Case (Chair), Fostering the development of number sense in the elementary and middle grades. Symposium conducted at the meeting of the American Educational Research Association, San Diego, CA.

Kalchman, M., & Case, R. (1999). Teaching mathematical functions in primary and middle school: An approach based on Neo-Piagetian theory. *Scientia Pedagogica Experimentalis*, *XXXV* (1), 7-53.

Kamii, C., & Dominick, A. (1997). To teach or not to teach algorithms. Journal of Mathematical Behavior, 16, 51-61.

Kitchener, K. S. (1983). Cognition, metacognition, and epistemic cognition: A three level model of cognitive processing. Human Development, 4, 222-232.

Kitchener, K. S. and King, P. A. (1981). Reflective judgment: Concepts of justification and their relationship to age and education. Journal of Applied Developmental Psychology, 2, 89-116.

Kuhn, D. (1991). The skills of argument. New York: Cambridge University Press.

Kuhn, D. (1992). Thinking as argument. Harvard Educational Review, 62, 155-178.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. Cognition and Instruction, 3, 305-342.

Lampert, M. (1992) Teaching and learning long division for understanding in school. In G. Leinhardt, R. Putnam, & R. Hattrup (Eds.), Analysis of arithmetic for mathematics teaching. Hillsdale, N.J.: Lawrence Erlbaum Associates.

Lampert, M., Rittenhouse, P., and Crumbaugh, C. (1996). Agreeing to Disagree: Developing Sociable Mathematics Discourse. In D. R. Olson and N. Torrance (Eds.), The Handbook of Education and Human Development. Cambridge: Blackwell.

Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. Review of Educational Research, 60(1), 1-64.

Madgidson, S. (1992). From the laboratory to the classroom: A technology-intensive curriculum for functions and graphs. Journal of Mathematical Behavior, 11,(4), 361-376.

Markovits, Z, Eylon, B, & Bruckheimer, M. (1986). Functions today and yesterday. For the Learning of Mathematics, 6, 2. 18-28.

McNeal, B. (1995). Learning not to think in a textbook-based mathematics class. Journal of Mathematical Behavior, 14, 205-234.

Moschovich, J. (1996). Moving up and getting steeper: Negotiating shared descriptions of linear graphs. Journal of the Learning Sciences, 5(3), 239-277.

Olson, D. & Bruner, J. (1996). Folk psychology and folk pedagogy. In D. Olson and N. Torrance (Eds.), The handbook of education and human development (pp. 9-27). Cambridge: Blackwell.

Olson, D. & Katz, S. (In press). The fourth folk pedagogy. In B. Torff and R. Sternberg (Eds.), Understanding and teaching the intuitive mind. Hillsdale, NJ: Erlbaum.

Ontario Ministry of Education and Training (1997). The Ontario Curriculum Grades 1-8: Mathematics. Toronto: Queen's Printer.

Perry, W. (1970). Forms of intellectual and ethical development in the college years. New York: Holt, Rinehart & Winston.

Schoenfeld, A., Smith, J, & .Arcavi, A. (1990). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), Advances in instructional psychology. Hillsdale, N.J.: Lawrence Erlbaum Associates.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy. West laFayette, IN: Mathematical Association of America. 25-58.

Skemp, R. (1976). Relational understanding and instrumental understanding. Mathematics Teaching, 77, 152-163.

Swan, M. (1982). The teaching of functions and graphs. In G. Van Barnveld & H. Krabbendam (Eds.), Conference on functions (Reference 1, pp. 151-164). Enschede, The Netherlands: Foundation for Curriculum Development.

Vakali, M. (1984). Childrens' thinking in arithmetic word problem solving. Journal of Experimental Education, 53, 106-113.